Comparison of numerical schemes for the 2 layer, 1 dimensional shallow water equations

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1 Introduction

There are a multitude of contrasting processes that occur simultaneously within the ocean at any given moment. Each of these processes act on their own particular time scale, which may differ wildly from each other. An extremely fast motion is sound waves which travel at approximately 1.5 km/s within the ocean. In comparison, the barotropic gravity wave speed in water depth of 1 km is $c_{\text{bt}} = \sqrt{gH} \approx 100$ m/s. The baroclinic gravity wave speed will be smaller still because of the small restoring force caused by a small density difference. A stable numerical scheme designed to describe each of these motions accurately would need to have sufficient temporal and spatial resolution.

The dimensionless number characterizing the necessary time step, Δt, for a given grid spacing, Δx, and velocity, c, is the Courant number, $C = c\Delta t/\Delta x$, where c is the speed of the fastest motion. For a scheme to be stable the Courant-Friedrichs-Lewy (CFL) condition is often stated as $C < 1$. Re-writing as $c\Delta t < \Delta x$ makes it clear that the distance a wave travels within one time step should not exceed the grid spacing. However, the CFL constraint may be relaxed if a particular numerical method is utilized, [1] as we shall see in section 3. This condition, however, is not a sufficient one and it is often necessary for C to be smaller than the prescribed criterion. Suppose, for now, that $C = 1$ is sufficient and that a numerical scheme has a grid spacing of 1 m. With these parameters the maximum theoretical time step necessary to accurately describe sounds waves is approximately 0.6 ms! A time step of this size will require an enormous number of steps to simulate a reasonable amount of time, the computational cost of which would be incredible and entirely unnecessary because of the small impact that sound waves have on the larger oceanic community.

This is an extreme example, but it clearly highlights the numerical issues which arise because of fast motions. Fortunately, this project is set within the context of Shallow Water Theory where sounds waves do not exist and there is no need to worry about resolving them. For the two layer model used within this project only the internal and surface gravity waves, or alternatively, motion from the barotropic and first baroclinic modes exist.

One extremely active field of research is the topic of internal waves. The oceanic ecosystem is dominated by internal dynamics, of which the surface motion has little effect. Furthermore, in comparison to surface waves, internal waves have larger amplitudes, contain more energy, and are able to more efficiently transport material. Therefore, I will focus on internal dynamics within the context of the two layer shallow water equations.

The physical schematic for two shallow layers of water used within this work is shown in Figure 1. The free surface displacement, η_1 is on average a depth H_1 above the internal surface, η_2, which is an average depth H_2 above the sea floor.

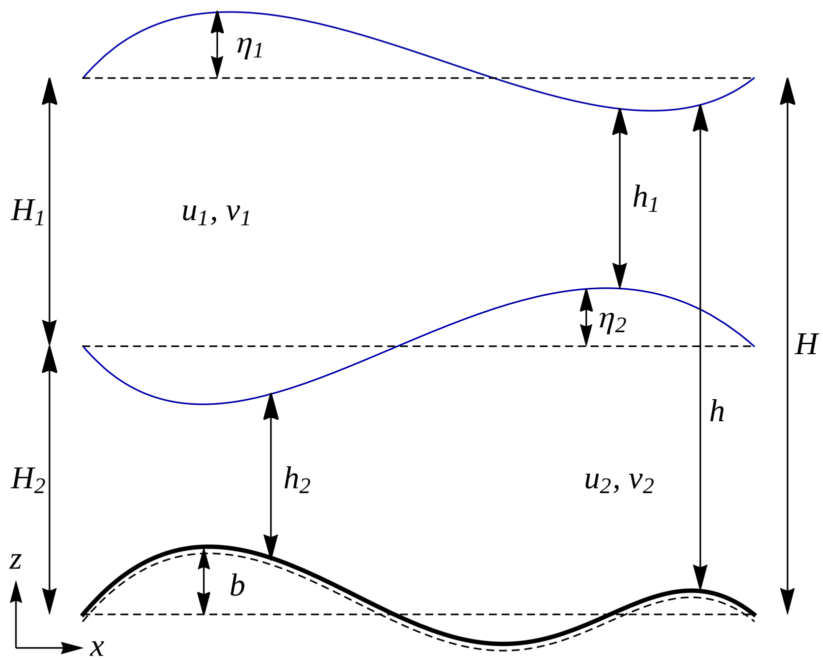


Figure 1: Schematic for two immiscible layers.

Table 1: Parameters used in numerical schemes.

L_x (km)	N_x	H_1 (m)	H_2 (m)	g (m/s^2)	$r = \rho_1/\rho_2$	g' (m/s^2)	f (Hz)
50	1024	20	80	9.81	0.95	$0.05g$	0

The parameters used within this project are listed in Table 1. The layer depths were chosen to correspond to a continental shelf where the total depth is 100 m and the pycnocline is centred near the surface. The reduced gravity is just slightly larger than physical. All methods are capable of handling non-zero Coriolis parameters, but for the discussion of this paper rotation will be neglected.

Before discussing the numerical methods used within this project, it is worthwhile to first investigate the difference in the time scales related to the barotropic and baroclinic motion. The barotropic and baroclinic velocities are

$$c_{\mathrm{bt}} = \sqrt{gH} \approx 30\mathrm{m/s}$$
$$c_{\mathrm{bc}} = \sqrt{g' \frac{H_1 H_2}{H_1 + H_2}} \approx 3\mathrm{m/s}$$

where it is clear that the barotropic velocity is an order magnitude faster than the baroclinic velocity. Therefore a numerical scheme which removes the barotropic motion will be much more efficient because it will not need to resolve the faster surface motion.

I will focus on three separate numerical schemes in this paper: explicit, modified Higdon, and the rigid lid approximation. The first two are capable of handling periodic bathymetry while the rigid lid code is currently built to only handle a flat bottom. All simulations discussed in this work will have a flat bottom so as to make effective comparisons.

2 Explicit scheme

The two layer shallow water equations are:

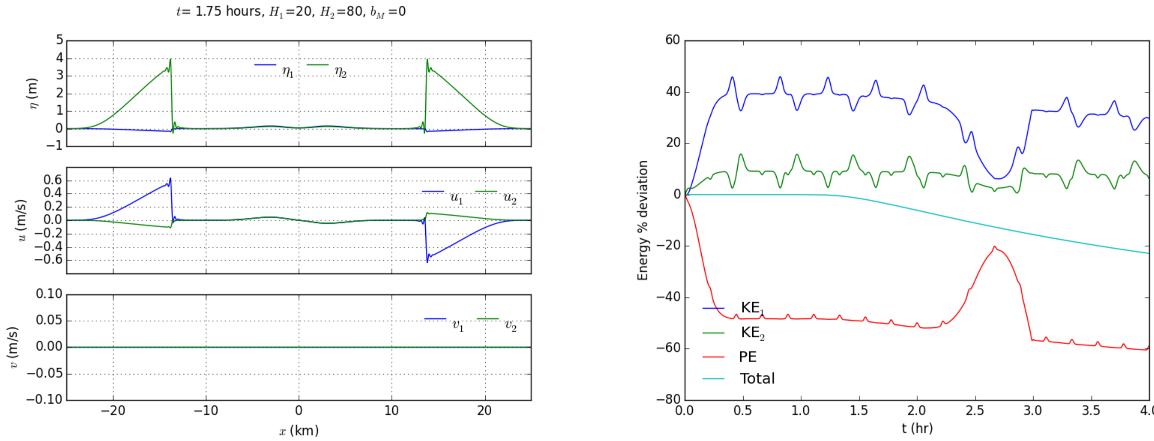
$$\partial_t (\eta_1 - \eta_2) + \partial_x [u_1 (H_1 + \eta_1 - \eta_2)] = 0 \tag{1a}$$
$$\eta_{2_t} + \partial_x [u_2 (H_2 + \eta_2 - b)] = 0 \tag{1b}$$
$$u_{1_t} + u_1 u_{1_x} - f v_1 + g \eta_{1_x} = 0 \tag{1c}$$
$$u_{2_t} + u_2 u_{2_x} - f v_2 + g \partial_x [(1-r)\eta_2 + r\eta_1] = 0 \tag{1d}$$
$$v_{1_t} + u_1 v_{1_t} + f u_1 = 0 \tag{1e}$$
$$v_{2_t} + u_2 v_{2_t} + f u_2 = 0 \tag{1f}$$

With a little work equation set 1 can be re-organized into something that looks like the conservative form for 1 layer:

$$h_{1_t} + \partial_x [\phi_1] = 0 \tag{2a}$$
$$h_{2_t} + \partial_x [\phi_2] = 0 \tag{2b}$$
$$\phi_{1_t} + \partial_x \left[\frac{\phi_1^2}{h_1} + \frac{1}{2} g h_1^2 \right] = -g h_1 \partial_x [h_2 + b] + f\theta_1 \tag{2c}$$
$$\phi_{2_t} + \partial_x \left[\frac{\phi_2^2}{h_2} + \frac{1}{2} g h_2^2 \right] = -g h_2 \partial_x [r h_1 + b] + f\theta_2 \tag{2d}$$
$$\theta_{1_t} + \partial_x \left[\frac{\phi_1 \theta_1}{h_1} \right] = -f\phi_1 \tag{2e}$$
$$\theta_{2_t} + \partial_x \left[\frac{\phi_2 \theta_2}{h_2} \right] = -f\phi_2 \tag{2f}$$

where $\phi_i = u_i h_i$ and $\theta_i = v_i h_i$ for $i = 1, 2$ are the layer's mass flux. Equations 2c and 2d are not conservative because of the coupling between the top and bottom layers, but the complete equation set is, as a whole, energy conserving.

The time stepping of equation set 2 is done with the third-order Adams-Bashforth method. To prepare for the AB3 stepping I have used an Euler step with a very small time-step before using



(a) Surface displacements and velocities at $t = 1.75$ h.

(b) Explicit scheme percentage energy deviation from the initial state.

Figure 2: Explicit Scheme.

an AB2 adaptive time-step scheme to bring the computed values onto an integer value of the AB3 time-step. The derivatives in equations 2 were computed using ffts, and an exponential filter with 50% cutoff and 90% cutoff of $0.6k_{\max}$ and $0.8k_{\max}$, respectively.

The initial variables for all the simulations and schemes were,

$$\eta_1(t=0) = u_1(t=0) = u_2(t=0) = 0$$
$$\eta_2(t=0) = h_m \exp\left(-(20x/L_x)^2\right)$$

with $h_m = 7$ m so as to force the baroclinic mode more than the barotropic mode. The initial displacement collapses upon itself forming two large baroclinic waves emanating away from the disturbance (Figure 2a). The collapse also forms barotropic waves, each with an amplitude of 15 cm. For an initial internal displacement of 7 m in a 20 m deep layer this is a very small disturbance, further emphasizing that the barotropic mode is inconsequential compared to the baroclinic mode.

As the simulation progresses, the internal waves quickly become steep because the layers are quite shallow. This is a standard result in shallow water theory and is fully expected. However, because derivatives are taken using finite discrete Fourier transforms the higher wavenumbers are not included in the differentiation. This, in conjunction with the spectral filter removing other high wavenumbers (so as to maintain a stable method) produces Gibbs oscillations at the back of the waves. This is an artifact of the chosen numerical method since the two layer shallow water equations are, as a whole, energy conserving. In reality, however, nothing conserves energy perfectly so the reduction of energy (Figure 2b) is in fact more realistic than the original formulation of the shallow water equations. This must be handled carefully because the energy is being removed at arbitrary wavelengths without any physical reasoning.

All this discussion on the spectral filtering is important because the the chosen physical configuration. Should the depths and width be increased by an order of magnitude, then the non-linear steepening would be reduced drastically and the model would appear to be much more conservative. Essentially the use of thin layers instead of thick ones have pushed the limitations of the numerical scheme.

3 Hidgon scheme

The explicit scheme can be modified slightly by time-stepping the layer depths before they are used in the momentum equations. Higdon [2] does something more complicated than this, but this is a first step in comparing schemes.

The idea is simple, equations 2a and 2b are computed to get h_1^{n+1} and h_2^{n+1}. which are then used in place of h_1^n and h_2^n in equations 2c-f. The aim is to increase the region of convergence so as to enable the use of a larger time-step.

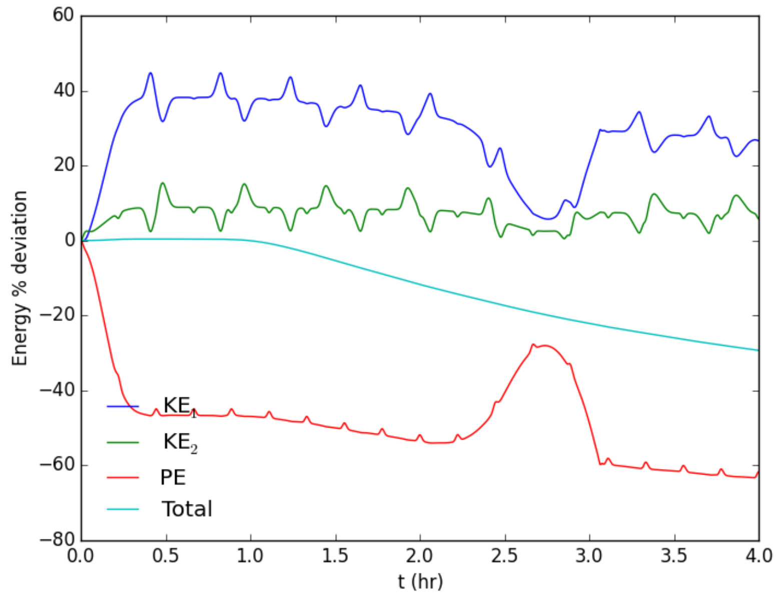


Figure 3: Higdon scheme percentage energy deviation from the initial state.

Both the explicit and Higdon schemes are very similar (compare Figure 2b and 3). However, the total energy percentage deviation after 4 hours is 29% in the Higdon scheme compared to 23% in the explicit scheme.

The major benefit to the Hidgon scheme is that the CFL constraint can be improved considerably. The maximum CFL constraint used in the explicit scheme is approximately 0.45 while in the Higdon scheme it can be as large as 1.4, roughly a factor of three increase. Use of the larger CFL constraint has little effect on the 29% energy error indicating that it is the numerical method which causes the deviation rather than the time-stepping.

One last minor difference between the Higdon and explicit schemes was the initial introduction of a small amount of energy (roughly 0.4%) into the system. It isn't much, but points to the need for further improvement upon this simplified method.

4 Rigid lid scheme

The explicit and Higdon scheme both included surface motions which had amplitudes on the order of 15 cm, while the internal waves had amplitudes of about 4 m. The ratio of amplitudes is approximately 0.04 emphasizing the disparity of their sizes and the insignificance of the surface waves. The rigid lid method removes surface motions by placing a hard lid where the free surface previously existed (Figure 4). The difficulty with this method is the extra work needed to find the pressure at the lid.

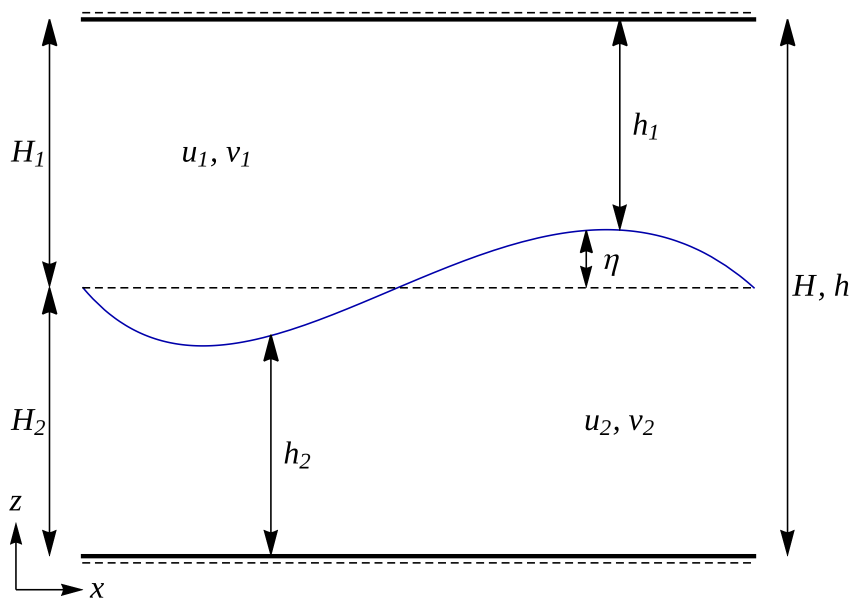


Figure 4: Schematic for two immiscible layers with a rigid lid.

Following the work by Derek Steinmoeller [3], the rigid lid approximation for the 2 layer shallow water equations in one dimension with a weakly non-hydrostatic pressure is

$$h_{1_t} + \partial_x [\phi_1] = 0 \tag{3a}$$

$$h_{2_t} + \partial_x [\phi_2] = 0 \tag{3b}$$

$$\phi_{1_t} + \partial_x \left[\frac{\phi_1^2}{h_1}\right] = -h_1 p_x + f\theta_1 \tag{3c}$$

$$\phi_{2_t} + \partial_x \left[\frac{\phi_2^2}{h_2}\right] = -h_2 p_x - g(1-r)h_2 h_{2_x} + \gamma\phi_{2_{xxt}} + f\theta_2 \tag{3d}$$

$$\theta_{1_t} + \partial_x \left[\frac{\phi_1\theta_1}{h_1}\right] = -f\phi_1 \tag{3e}$$

$$\theta_{2_t} + \partial_x \left[\frac{\phi_2\theta_2}{h_2}\right] = -f\phi_2 \tag{3f}$$

where p is the lid pressure, and $\gamma\phi_{2_{xxt}}$ is the weakly non-hydrostatic pressure which may also be called the auxiliary pressure. The origin of this term is not important to the discussion of this paper, but it originates in the desire to have the correct dispersion relation. To make proper comparison to the previous two numerical schemes this term will be turned off, but it is added here since it was not too much work to add into the numerical scheme and a future direction could be to add a similar term into the previous two methods.

The main difference the rigid lid places on the numerical scheme is that there is no evolution equation for the pressure. This pressure must be calculated at each time step and could be computationally taxing. Here I will outline the steps to calculate the new layer depths and mass fluxes.

Integrating equation set 3 from $t = n\Delta t$ to $t = (n+1)\Delta t$ gives

$$h_1^{n+1} = h_1^n + \Delta t R_{h_1}^n \tag{4a}$$
$$h_2^{n+1} = h_2^n + \Delta t R_{h_2}^n \tag{4b}$$
$$\phi_1^{n+1} = \phi_1^n + \Delta t R_{\phi_1}^n - \Delta t h_1 p_x^n \tag{4c}$$
$$\phi_2^{n+1} = \phi_2^n + \Delta t R_{\phi_2}^n - \Delta t h_2 p_x^n + \Delta t \gamma \phi_{2_{xxt}}^n \tag{4d}$$
$$\theta_1^{n+1} = \theta_1 + \Delta t R_{\theta_1}^n \tag{4e}$$
$$\theta_2^{n+1} = \theta_2 + \Delta t R_{\theta_2}^n \tag{4f}$$

where $\vec{R}^n = 1/12\left(23\vec{F}^n - 16\vec{F}^{n-1} + 5\vec{F}^{n-2}\right)$ after the AB2 and Euler steps have been initially calculated at the beginning of the simulation. With this definition we have,

$$\vec{F}^n = \begin{pmatrix} -\partial_x\left[\phi_1^n\right] \\ -\partial_x\left[\phi_2^n\right] \\ -\partial_x\left[\frac{(\phi_1^n)^2}{h_1^n}\right] + f\theta_1^n \\ -\partial_x\left[\frac{(\phi_2^n)^2}{h_2^n}\right] - g(1-r)h_2^n h_{2_x}^n + f\theta_2^n \\ -\partial_x\left[\frac{\phi_1^n\theta_1^n}{h_1^n}\right] - f\phi_1^n \\ -\partial_x\left[\frac{\phi_2^n\theta_2^n}{h_2^n}\right] - f\phi_2^n \end{pmatrix}$$

The first two terms on the right hand side of equation set 4 are evaluated first to produce predicted layer depths and mass fluxes. That is,

$$\vec{V}^\dagger = \vec{V}^n + \Delta t \vec{R}^n$$

where $\vec{V}^n = (h_1^n, h_2^n, \phi_1^n, \phi_2^n, \theta_1^n, \theta_2^n)$. This now leaves us with,

$$h_1^{n+1} = V_{h_1}^\dagger \tag{5a}$$
$$h_2^{n+1} = V_{h_2}^\dagger \tag{5b}$$
$$\phi_1^{n+1} = V_{\phi_1}^\dagger - \Delta t h_1 p_x^n \tag{5c}$$
$$\phi_2^{n+1} = V_{\phi_2}^\dagger - \Delta t h_2 p_x^n + \Delta t \gamma \phi_{2_{xxt}}^n \tag{5d}$$
$$\theta_1^{n+1} = V_{\theta_1}^\dagger \tag{5e}$$
$$\theta_2^{n+1} = V_{\theta_2}^\dagger \tag{5f}$$

It is clear that the predicted layer depths and meridional mass flux are the correct values for the new time. We are now left with the choice to compute the lid pressure or the auxiliary pressure. The lid pressure ensures that the scheme is incompressible, so it's computation will be left till after the auxiliary pressure is found and incorporated into the the predicted mass fluxes.

Taking the x-derivative of equation 3d, ignoring the lid pressure and letting $\lambda = \phi_{2_{xt}}$ gives,

$$\lambda = F_{\phi_2} + \gamma \lambda_{xx}$$

This is easily solved as to give

$$\phi_{2_{xxt}} = \lambda_x = \text{ifft}\left\{ \frac{(ik)^2}{1-\gamma(ik)^2} \hat{F}_{\phi_2} \right\}$$

Equations 5c and 5d are now updated to give

$$\phi_1^{n+1} = V_{\phi_1}^* - \Delta t h_1 p_x^n \tag{6a}$$
$$\phi_2^{n+1} = V_{\phi_2}^* - \Delta t h_2 p_x^n \tag{6b}$$

where $V_{\phi_1}^* = V_{\phi_1}^\dagger$, and $V_{\phi_2}^* = V_{\phi_2}^\dagger + \Delta t \gamma \lambda_x^n$. Taking the divergence (that is, ∂_x) of the sum of equations 6 gives,

$$\partial_x \left[h_1^{n+1} u_1^{n+1} + h_2^{n+1} u_2^{n+1} \right] = \partial_x \left[V_{\phi_1}^* + V_{\phi_2}^* \right] - \Delta t h p_{xx}^n$$

Since the lid is a fixed boundary the baroclinic transport is zero,

$$\partial_x \left[h u_{\text{bt}} \right] = \partial_x \left[V_{\phi_1}^* + V_{\phi_2}^* \right] - \Delta t h p_{xx}^n$$

and the left hand side is zero because the fluid is conservative. Therefore, the elliptic problem for pressure is,

$$p_{xx}^n = \frac{1}{\Delta t h} \partial_x \left[V_{\phi_1}^* + V_{\phi_2}^* \right]$$

This can be solved as

$$p^n = \text{ifft}\left\{ \frac{1}{ik\Delta t h} \text{fft}\left[V_{\phi_1}^* + V_{\phi_2}^* \right] \right\}$$

The absolute pressure is unimportant to the evolution of the momentum equations so the gradient of lid pressure is

$$p_x^n = \text{ifft}\left\{ \frac{1}{\Delta t h} \text{fft}\left[V_{\phi_1}^* + V_{\phi_2}^* \right] \right\} = \frac{1}{\Delta t h} \left(V_{\phi_1}^* + V_{\phi_2}^* \right) \tag{7}$$

which is then used in equation set 6. This simple relation between the predicted zonal mass fluxes and the lid pressure is only capable because the bathymetry is flat. Would the bathymetry be be non-constant then the first definition of equation 7 would have to be used.

The energy is still non-conservative because of the filtering. The maximum energy deviation is slightly worse than the Higdon scheme at 30%.

5 Discussion

The motivation of this work was to find a numerical scheme which is efficient at accurately resolving the internal motion of the two layer shallow water system. Table 2 highlights the differences in the speed of each of the discussed methods. The slowest method was the explicit scheme which required many time steps, N_t, to remain stable. It also required the smallest CFL condition which

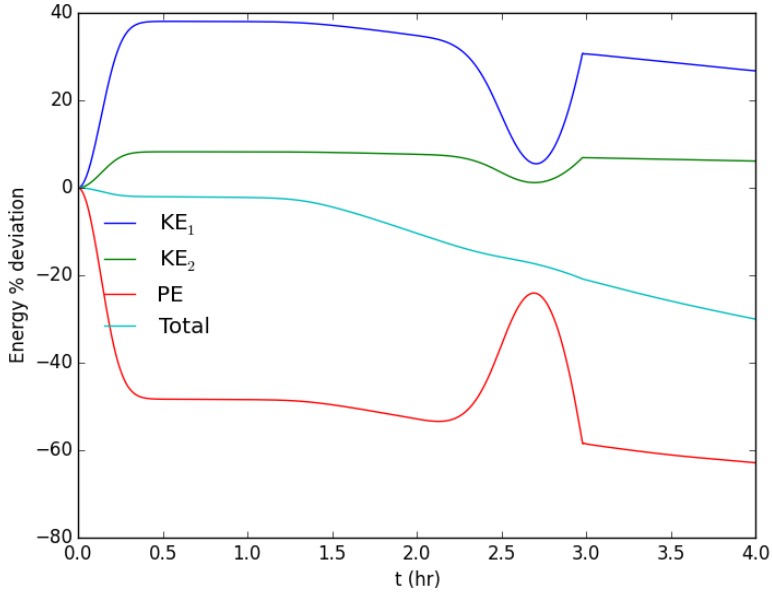


Figure 5: Rigid lid percentage energy deviation from the initial state. Auxiliary pressure not included.

contributed to the small time step. The Hidgon scheme was roughly 2.5 times faster because of a relaxed CFL condition.

The python profiler was used to measure the time the code runs apart from the plotting and filtering that is needed (time per loop). Both the explicit and Higdon scheme had similar calculation time per loop (1 ms), while the rigid lid was three times larger. The rigid lid was the fastest because of the large time-step afforded by ignoring the surface motion.

Table 2: Temporal comparison of schemes.

Method	N_t	CFL	Δt (s)	Comp. time (s)	Time per loop (ms)
Explicit	23,092	0.4	0.6	52	1.0
Higdon	6,597	1.4	2.2	20	1.1
Rigid Lid	1,836	0.45	7.8	11	3

There are a few other numerical techniques that could also be included in this comparison. One common method for increasing the time-step is to use an implicit scheme. This would be similar to the rigid lid in that the time-step could be larger, but the computational cost would also increase because of the need to solve a matrix inversion. Another method discussed by Duran [1] is the semi-implicit scheme which treats some terms as an average between the value at t^{n+1} and t^n. Again, this would require more computation to solve the matrix inversion. The semi-implicit method was written, but had some bugs and required more work to make it useful for comparison to the other methods listed above.

References

[1] D. R. Durran. *Numerical methods for fluid dynamics: with applications to Geophysics*. Springer, 2nd edition, 2010.

[2] Robert L. Higdon. A two-level time-stepping method for layered ocean circulation models: Further development and testing. *J. Comput. Phys.*, 206(2):463–504, July 2005.

[3] D. Steinmoeller. *High order numerical methods in lake modelling*. PhD thesis, University of Waterloo, 2014.